Dynamic Utilities

and Stochastic Differential Equations

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Some History

Generalities

- Since Bernoulli : Paradox of St Petersbourg
  - Dou you prefer 1 dollar today
  - or play to a lottery with gain 100 dollars
- Basically, Bernoulli assumed that the value given to a particular wealth amount depends on its relative importance to total wealth.
- The utility function is a function of capital that can associate a certainty equivalent to a given bet, as it is indifferent to take the bet or its certainty equivalent.
- Logarithmic utility $u(x) = \log x$

Regular Utility Function

- A regular utility function u is a (positive) function defined on $[0, \infty)$
  - Concave : $\mathbb{E}(u(X)) \leq u(\mathbb{E}(X))$
  - Increasing
  - Inada condition : $u(x)$ is a $\mathcal{C}^2$-function with marginal utility $u_x(.)$, decreasing from $+\infty$ to 0.
- Convex Conjugate Utility $\tilde{u}$
  - $\tilde{u}$ is the Fenchel transform of $-u(. - x)$
  - Under Inada condition, $\tilde{u}(y) = \sup_{x>0} \big(u(t,x) - x\,y\big)$
  - The optimum is achieved at $u'_x(x^*) = y$, and $-\tilde{u}_y = (u_x(.))^{-1}$
  - $\tilde{u}(y) = u(-\tilde{u}'_y(y)) + y\tilde{u}'_y(y)$
- Certainty equivalent : $\mathbb{E}(u(X)) = u(c(X))$ concavity $\Longrightarrow c(X) \leq \mathbb{E}(X)$

Risk Aversion coefficient

Quantities of interest

- Risk Aversion coefficient $\alpha(x) = -\frac{u_{xx}(x)}{u_x(x)}$, relative $\hat{\alpha}(x) = -\frac{xu_{xx}(x)}{u_x(x)}$,
- Risk tolerance coefficient $\tau(x) = (\alpha(x))^{-1}$
- Typical example : power utility For $\alpha \in (0,1)$,
  $u(x) = \frac{x^{1-\alpha}}{1-\alpha}$ with conjugate $-\tilde{u}_y(y) = -y^{-1/\alpha}$

Basic optimization problem

- for given $\mathscr{X}$ convex family of random variables X_T, and some state price density Y_T with $\mathbb{E}(Y_T) \leq 1$,

$$\max \mathbb{E}(u(X))|X \in \mathscr{X} \text{ , with budget constraint } \mathbb{E}(Y_T X_T) \leq x$$

- Solution via duality : Lagrange multiplier technics
  - The problem is equivalent to : $\max\{\mathbb{E}(u(X) + y(x - Y_T X_T)|X \in \mathscr{X}\}$
  - If $-\tilde{u}_y(yY_T) \in \mathscr{X}$, then optimum is $X_T^* = -\tilde{u}_y(yY_T)$
  - y is selected by achieved the budget constraint, if it is possible

$$\mathbb{E}[-\tilde{u}_y(yY_T)Y_T) = x$$

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Performance measurement in debate

Performance and risk measurement are fundamental in mathematical finance, risk-management and portfolio optimization

An old new question

- For a long time, expected utility has been the standard for dynamic risk
- Extended into a robust formulation by taking into account ambiguity on the ”reference probability measure” by min-max point of view

$$\max_{X_T \in \mathcal{X}} \min_{Q \in \mathcal{Q}} \mathbb{E}_Q[u(X_T)]$$

In relation with risk measures (Foellmer, Schied)

Dynamic point of view

Dynamic view

- Expected utility is overly restrictive in expressing reasonable risk aversion in temporal setting
- Intertemporal substitution and risk aversion are inflexibly linked
- Stochastic Differential Utility (Duffie, Epstein, Skiadas...) : the local variation is depending on the expected future utility ;
- BSDE's point of view

$$-dU_t(\xi_T) = g(t, U_t, Z_t)dt - Z_t dW_t, \quad U_T(\xi_T) = u(\xi_T) \text{ for given } u$$

- For a given terminal utility function u, A solution consists into two processes
  - The progessive utility $U_t(\xi_T)$
  - The progressive diffusion coefficient Z_t

Investment Banking and Utility Theory

Remarks and Comments from M.Musiela, T.Zariphopoulo (2002-2009)

- Classical or recursive utilities are defined in isolation to the investment opportunities given to agent
- The investor may want to use intertemporal diversification, I.e. implement short, medium, long term strategies
- Need of intertemporal consistency of optimal strategies. Can the same utility function be used for all time horizon ?
- At the optimum the investor should become indifferent to the investment horizon.

+ C.Rogers +Berier+Tehranchi, Henderson-Obson, Zitkovic (2002-2011)

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Progressive Utility

Definition of Progressive Utility

- A progressive utility is a positive family $\mathbf{U} = \{U(t,x) : t \geq 0, x > 0\}$
  - Progressivity : for any $x > 0$, $t \mapsto U(t,x)$ is a progressive random field
  - Concavity : for $t \geq 0$, $x > 0 \mapsto U(t,x)$ is an increasing concave function.
  - Inada condition : $U(.,x)$ is a $\mathcal{C}^2$-function with marginal utility $U_x(.,.)$, decreasing from $+\infty$ to 0.
  - Initial condtion : u a deterministic positive $\mathcal{C}^2$-utility function with Inada condition

Convex Conjugate Dynamic Utility $\tilde{U}$

- $\tilde{U}$ is the Fenchel transform of $-U(.-x)$
  - Under Inada condition, $\tilde{U}(t,y) = \sup_{x>0, x\in Q^+} \left(U(t,x) - x\,y\right)$
  - The optimum is achieved at $U'_x(t,x^*) = y$, and $-\tilde{U}'_y(t,) = (U'_x(t,.))^{-1}(y)$
- $\tilde{U}(t,y) = U(t, -\tilde{U}'_y(t,y)) + y\tilde{U}'_y(t,y)$

Consistent Dynamic Utility

Let $\mathscr{X}$ be a convex family of non negative portfolios, called Test porfolios

An $\mathscr{X}$-consistent dynamic utility $U(t,x)$ is a progressive utility s.t

- Consistency with the family of test portfolios

  For any admissible wealth process $X \in \mathscr{X}$, $\mathbb{E}(U(t,X_t)) < +\infty$ and

  $$\mathbb{E}(U(t,X_t)/\mathcal{F}_s) \leq U(s,X_s), \ \forall s \leq t.$$

- Existence of optimal For any initial wealth $x > 0$, there exists an optimal wealth process (benchmark) $X^* \in \mathscr{X}$, $X^*_0 = x$,

  $$U(s,X^*_s) = \mathbb{E}(U(t,X^*_t)|\mathcal{F}_s) \ \forall s \leq t.$$

In short for any admissible wealth $X \in \mathscr{X}$, $U(t,X_t)$ is a supermartingale, and a martingale for the optimal benchmark X^*.

A General Market Model I

Incomplete Market : Let W be a n-Brownian motion, a short rate process r_t and a risk premium vector η_t, and $\mathscr{X}$ the class of (positif) wealth processes X^κ driven by the self-financing equation

$$dX_t^\kappa = X_t^\kappa \big[r_t dt + \kappa_t.(dW_t + \eta_t^\sigma dt) \big], \; \eta_t^\sigma, \kappa_t \in \mathcal{R}_t^\sigma$$

- σ_t is the dxn volatility matrix, and $\sigma_t.\sigma_t^\top$ is invertible.
- Let π_t be the wealth proportions invested in the different assets, and $\kappa_t = \sigma_t \pi_t$,
- Constraints : $\mathcal{R}_t^\sigma$ is a family of adapted subvector spaces in $\mathbb{R}^n$, typically $\mathcal{R}_t^\sigma = \sigma_t(\mathbb{R}^d)$, $d \leq n$.

A General Market Model II

- $\eta_t^\sigma \in \mathcal{R}_t^\sigma$ defined as the projection of η_t on $\mathcal{R}_t^\sigma$ is the minimal risk premium,

  All processes are adapted with good integrability properties

Def A process Y is said to be a state price density or (adjoint process) if for any $\kappa \in \mathcal{R}^\sigma$, $Y_. X_.^\kappa$ is a local martingale $\Rightarrow$ there exists $\nu \in \mathcal{R}^{\sigma,\perp}$:

$$\frac{dY_t^\nu}{Y_t^\nu} = -r_t dt + (\nu_t - \eta_t^\sigma).dW_t, \ \nu_t \in \mathcal{R}_t^{\sigma,\perp}$$

Value Function of Classical Utility Problem

Classical problem : Backward point of view

- Given a utility function $u(T,x)$ at given time horizon T, the problem at time r is to maximize over all admissible portfolios starting from (r,x), the conditional expected utility of the terminal wealth,

$$V(r,x,(u,T)) = \operatorname{ess\,sup}_{X\in\mathcal{X}(r,x)} \mathbb{E}(u(T,X_T)/\mathcal{F}_r)$$

- Dynamic programming principle

$$V(t,X_t,(u,T)) = V(t,X_t,(V(t+h,.,(u,T)),t+h)), a.s.$$

- Maximum principle $\Longrightarrow$ Comparison theorem $\Longrightarrow$ concavity of $V(r,x)$.
- Conc : $V(t,x,(u,T)) = \operatorname{ess\,sup}_{X\in\mathcal{X}(t,x)} \mathbb{E}[V(t+h,X_{t+h},(u,T))|\mathcal{F}_t]$ is a consistent progressive utility, with initial value $v(0,x) = V(0,x,(u,T))$

Progressive Utility of Itô Type

- Assume the progressive utility U to be a family of Itô semimartingales with local characteristics (β, γ) (β=drift, γ= diffusion)

$$dU(t,x) = \beta(t,x)dt + \gamma(t,x)dW_t$$

- Assume the conjugate progressive utility $\tilde{U}$ to be also of Itô type.

$$d\tilde{U}(t,x) = \tilde{\beta}(t,x)dt + \tilde{\gamma}(t,x)dW_t$$

Open questions at this stage

- Under which assumptions on (β, γ) the solution is concave and increasing,
- What kind of relationship between (β, γ) and $(\tilde{\beta}, \tilde{\gamma})$?
- Under which assumptions on (β, γ) only, $\tilde{U}$ is also of Itô type
- Main difficulties come from the forward definition : Absence of maximum principle or comparison theorem.

Consistent Dynamic Utilities

- Assume U to be $\mathscr{X}$-consistent. How express on (β,γ) the supermartingale property of $U(t, X^{\kappa}_{.})$
- Is the convex conjugate utility associated with the same kind of optimization problem ?
- Existence of optimal solutions ?
- In the classical backward framework,
  - By maximum principle, $U'_x(t, X^*_t(x)) = Y^*_t(u'_x(x))$.
  - $Y^*_{.}(y)$ is the optimal solution of the dual problem

Open questions

- Is these properties still hold true
- Regularity of $X^*_t(x)$ and $Y^*_t(y)$ with respect of their initial condition ?
- If $X^*_t(x)$ is monotone, $U'_x(t,x)) = Y^*_t(u'_x((X^*_t(x))^{-1}))$?

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Concavity and Stochastic DIfferential Equation

Concavity and SDE

Let us consider a progressive differentiable random field $\mathbf{U}$, such that $\mathbf{U}$ and $\mathbf{U_x}$ are Itô random fields with local characteristics (β, γ) and (β_x, γ_x).

(i) NECESSARY CONDITION If U is a progressive utility with conjugate $\tilde{U}$. Then $U_x(t,.)$ is decreasing in x from ∞ to 0, with inverse$-\tilde{U}_y(t,.)$.

$$dU_x(t,.x) = \beta_x(t,x)dt + \gamma_x(t,x).dW_t$$

(ii) Intrinsic SDE Then $U_x(.,x) = Z_.(u_x(x))$, where

- $Z_.(z)$ is a strong solution of the following intrinsic SDE,

$$dZ_t = \mu(t,Z_t)dt + \sigma(t,Z_t)\,dW_t, \quad Z_0 = z$$

- with coefficients $\mu(t,z) = \beta_x(t,-\tilde{U}_y(t,z)),\ \sigma(t,z) := \gamma_x(t,-\tilde{U}_y(t,z)$ with $\mu(t,0)=0,\ \sigma(t,0)=0$
- which is increasing and differentiable on z with range $(0,\infty)$.

Utility and Primitive of Intrinsic SDE

$$dZ_t = \mu(t, Z_t)dt + \sigma(t, Z_t)dW_t,\ Z_0 = z$$

Characterization as primitive of monotone SDE

If the SDE has a unique strong solution $Z_.(z)$, increasing and differentiable in z from 0 to ∞,

- For any utility u, $Z_t(u_x(x))$ is positive, decreasing progressive random field, with range $(\infty, 0)$.
- If $Z_.(u_x(x))$ is integrable in a neighborhood of $x = 0$, the primitive $\{U(t,x) = \int_0^x Z_t(u_x(z))dz, t \geq 0, x > 0\}$ is a progressive utility.

SDE with random coefficients

Protter , Kunita books

Lipschitz condition Let the one-dimensional SDE,

$$dZ_t = \mu(t, Z_t)dt + \sigma(t, Z_t)dW_t,$$

- Assume there exists C_t and K_t with $\int_0^T (C_t + K_t^2)dt < +\infty$.
- Assume that $\mu(t, 0) \equiv 0, \quad \sigma(t, 0) \equiv 0$. and

$$|\mu(t,x,\omega)-\mu(t,y,\omega)| \leq C_t(\omega)|x-y|, ||\sigma(t,x,\omega)-\sigma(t,y,\omega)|| \leq K_t(\omega)|x-y|$$

- Then, for any $z \in \mathbb{R}_+$ there exists a unique strong solution Z^z of the SDE increasing with respect to its initial condition $Z_0 = z$.
- The range of the map $z \mapsto Z(., z)$ is $]0, +\infty[$ and $Z(., z)$ is integrable near to 0 and to infinity.

Applications to Progressive Utility

SUFFICIENT CONDITIONS If there exist random Lipschitz bounds C_t and K_t^2 integrable in time such that *a.s*,

$$||\beta_x(t,x) \leq C_t\,|U_x(t,x)|, \quad ||\gamma_x(t,x)|| \leq K_t\,|U_x(t,x)|$$

$$|\beta_{xx}(t,x)| \leq C_t\,|U_{xx}(t,x)|, \quad ||\gamma_{xx}(t,x)|| \leq K_t\,|U_{xx}(t,x)|$$

Then the derivatives of the coefficients μ_x and σ_x are spatially bounded, then the SDE has unique strong solution and U is a progressive utility.

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Itô's Ventcell Formula

For technical regularity problems see books of Kunita, or Carmona-Nualart.

- The identity $\tilde{U}(t,y) = U(t, -\tilde{U}_y(t,y)) + y\tilde{U}_y(t,y)$ is based on the $\mathcal{C}^2$ random field U along the random process $-\tilde{U}_y(t,y)$. Need to extension of the Itô's formula.
- Itô's Ventcell Formula Let $F(t,x)$ be a $\mathcal{C}^2$ Itô random field (β,γ), such that $F_x(t,x)$ is associated with (β_x,γ_x). For any Itô semimartingale X,

$$\begin{aligned} dF(t,X_t) &= \beta(t,X_t)dt + \gamma(t,X_t).dW_t \\ &+ F_x(t,X_t)dX_t + \frac{1}{2}F_{xx}(t,X_t)\langle dX_t\rangle + \langle dF_x(t,x), dX_t\rangle|_{x=X_t} \end{aligned}$$

Conjugate utility dynamics

- Apply this result to $F(t,x) = U(t,x) + xy$ with $X_t = -\tilde{U}_y(t,y)$ (assumed to be Itô), by observing that $F_x(t,x) = 0$ when $x = -\tilde{U}_y(t,y)$.
- Dynamics of the conjuguate utility Assume $(U, \tilde{U})$ with characteristics (β,γ) and $(\tilde{\beta},\tilde{\gamma})$ and $(U_x, \tilde{U}_y)$ associated with the derivatives.

$$\begin{aligned} d\tilde{U}(t,y) &= \gamma(t,-\tilde{U}_y(t,y)).dW_t + \beta(t,-\tilde{U}_y(t,y))dt \\ &+ \frac{1}{2}\tilde{U}_{yy}(t,y)\|\gamma_x(t,-\tilde{U}_y(t,y))\|^2\,dt \end{aligned}$$

Marginal Conjuguate Utility

- Dynamics of the marginal conjuguate utility
  Let (μ, σ) be the random coefficients of the SDE associated with U_x
  $\mu(t,z) = \beta_x(t, -\tilde{U}_y(t,z)),\ \sigma(t,z) := \gamma_x(t, -\tilde{U}_y(t,z)$
- Define $\tilde{L}^{\sigma,\mu}$ to be the adjoint operator,
  $\tilde{L}^{\sigma,\mu} = \frac{1}{2}\partial_y(||\sigma(t,y)||^2\partial_y) - \mu(t,y)\partial_y$.
- Then the inverse of $-U_x$, $\tilde{U}_y$ is a monotonic solution of the SPE, with initial condition $\tilde{U}_y(0,y) = \tilde{u}_y(y)$

Change of variable SPDE

$$dG(t,y) = -G_y(t,y)\sigma(t,y).dW_t + \tilde{L}^{\sigma,\mu}(G)(t,y)dt$$

- Other application : dynamic copula

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Drift Constraint

Let U be a Itô-Ventzel regular utility and $X^{\kappa}_{\cdot}$ an admissible wealth

$$dU(t,x) = \beta(t,x)dt + \gamma(t,x)dW_t, \quad dX^{\kappa}_t = X^{\kappa}_t[r_t dt + \kappa_t.(dW_t + \eta^{\sigma}_t dt)],$$

Itô-Ventcel Formula

$$dU(t,X^{\kappa}_t) = \beta(t,X^{\kappa}_t)dt + \gamma(t,X^{\kappa}_t).dW_t + \langle \gamma_x(t,X^{\kappa}_t), X^{\kappa}_t \kappa_t \rangle dt.$$
$$+U_x(t,X^{\kappa}_t)X^{\kappa}_t \kappa_t dX_t + \Big(U_x(t,X^{\kappa}_t) r_t X^{\kappa}_t + \frac{1}{2}U_{xx}(t,X^{\kappa}_t)(X^{\kappa}_t)^2\|\kappa_t\|^2\Big)dt$$

HJB type constraints

$$dU(t,X^{\kappa}_t) = \big(U_x(t,X^{\kappa}_t)X^{\kappa}_t \kappa_t + \gamma(t,X^{\kappa}_t)\big).dW_t$$
$$+\Big(\beta(t,X^{\kappa}_t) + U_x(t,X^{\kappa}_t) r_t X^{\kappa}_t + \frac{1}{2}U_{xx}(t,X^{\kappa}_t)\mathcal{Q}(t,X^{\kappa}_t,\kappa_t)\Big)dt,$$

where $\mathcal{Q}(t,x,\kappa) := \|x\kappa\|^2 + 2x\kappa.\big(\dfrac{U_x(t,x)\eta^{\sigma}_t + \gamma_x(t,x)}{U_{xx}(t,x)}\big).$

Verification Theorem

- Verification Theorem Let γ_x^σ be the orthogonal projection of γ_x on $\mathcal{R}^\sigma$; and $\mathcal{Q}^*(t,x) = \inf_{\kappa\in\mathcal{R}^\sigma} \mathcal{Q}(t,x,\kappa)$;
  The minimum of this quadratic form is achieved at the optimal policy κ^*
  - $x\kappa_t^*(x) = -\frac{1}{U_{xx}(t,x)}\left(U_x(t,x)\eta_t^\sigma + \gamma_x^\sigma(t,x)\right)$
  - $x^2\mathcal{Q}^*(t,x) = -\frac{1}{U_{xx}(t,x)^2}||U_x(t,x)\eta_t^\sigma + \gamma_x^\sigma(t,x))||^2 = -||x\kappa_t^*(x)||^2$
- Drift constraint $\beta(t,x) = -U_x(t,x)r_t x + \frac{1}{2}U_{xx}(t,x)\|x\kappa_t^*(t,x)\|^2$
- Volatility The volatility $\gamma(t,x)$ verifies
  $U_x'(t,x)\eta_t^\sigma + \gamma_x'(t,x) = -xU_{xx}''(t,x)\kappa_t^*(x) - \nu^\perp(t,x) : \nu^\perp(t,x) \in \mathcal{R}_t^{\sigma,\perp}$
- Decreasing utility When $\gamma(t,x) \equiv 0$, classical optimal strategy
  $U_x'(t,x)\eta_t^\sigma = -U_{xx}''(t,x)x\kappa_t^*(x)$.

Utility Stochastic PDE

Optimal Wealth

- If $\kappa^*(t,x)$ is sufficiently smooth so that $\forall x > 0$ the equation

$$dX_t^* = X_t^*\big[r_t dt + \kappa_t^*(X_t^*).(dW_t + \eta_t^\sigma dt)\big]$$

has at least one positive solution X^*, then $U(t,X_t^*)$ is a local martingale.

- if the local martingale $\big(U(t,X_t^*)\big)_{t\geq 0}$ is a martingale, then the progressive utility U is a $\mathscr{X}$-consistent stochastic utility with optimal wealth process X^*.
- The semimartingale $U_x(t,X_t^*)$ is a state price density process, $dU_x(t,X_t^*) = U_x(t,X_t^*)\big[-r_t dt + \big(\eta_t^{U,\perp}(t,X_t^*) - \eta_t^\sigma\big)\big).dW_t\big]$
  where $\eta_t^{U,\perp}(t,x) = \frac{\gamma_x^\perp}{U_x}(t,x)$ is the orthogonal utility risk premium

Convex conjugate SPDE

Conjugate SPDE

- Let $\widetilde{U}$ be the conjugate of U, with Itô-Ventzel regularity, then

$$\begin{aligned} d\widetilde{U}(t,y) &= \overline{\beta}(t,-\tilde{U}'_y(t,y))dt + \gamma(t,-\tilde{U}'_y(t,y))dW_t \quad \text{where} \\ \overline{\beta}(t,x) &= \beta(t,x) - \frac{1}{2}\frac{\|\gamma'_x(t,x)\|^2}{U''_{xx}(t,x)} \end{aligned}$$

- $\overline{\beta}(t,x)$ is the solution of a minimization program achieved by the projection of $-\eta^\sigma_t\, U'_x(t,x) - \gamma'_x(t,x)$ on $(\mathcal{R}^\sigma)^\perp$, defined before as $\nu^\perp(t,x)$
- In new variable, $\tilde{\gamma}(t,y) = \gamma(t,-\tilde{U}'_y(t,y)), \tilde{\beta}(t,y) = \overline{\beta}(t,-\tilde{U}'_y(t,y))$

$$\tilde{\beta}(t,y) = r_t y \tilde{U}'_y(t,y) + \frac{-1}{2\tilde{U}''_{yy}}\Big(\|(-\eta^\sigma_t y \tilde{U}''_{yy} + \tilde{\gamma}'^{,\sigma}_y)\|^2 - \|\tilde{\gamma}'_y\|^2\Big)(t,y)$$

Convex consistent dual utility

Consistent Conjuguate Utility

Under previous assumption,

- The conjugate utility $\tilde{U}(t,y)$ is a convex decreasing stochastic flow,
- consistent with the family $\mathscr{Y}$ of semimartingales Y^ν, defined from

$$dY^\nu_t = Y^\nu_t[-r_t dt + (\nu_t - \eta^\sigma_t)dW_t, \quad \nu_t \in (\mathcal{R}^\sigma_t)^\perp]$$

- There exists a dual optimal choice $\tilde{\nu}^*(t,y) = \nu^\perp(t, -\tilde{U}'_y(t,y))$
- From any $y > 0$, the optimal dual process $Y^*_t(y) = Y^{\tilde{\nu}^*}_t(y)$ satisfies $Y^*_t(u'_x(x)) = U'_x(t, X^*_t(x))$
- If $X^*_t(x)$ is strictly monotone in x, by taking its inverse $\mathcal{X}(t,x)$, we obtain that $U'_x(t,x) = Y^*_t(u_x((\mathcal{X}(t,x)))$.

  No trivial calculation via stochastic calculus method.

New parametrization of the SPDE

Volatility versus optimal strategies

- There is a one to one correspondence between the derivative of the volatility γ_x and the optimal strategies κ^* and $\nu^{*,\perp}$

$$\begin{aligned}\gamma_x^\sigma(t,x) &= -U_{xx}(t,x)x\kappa^*(t,x) - U_x(t,x)\eta_t^\sigma \\ \gamma_x^\perp(t,x) &= U_x(t,x)\nu^{*,\perp}(t,U_x(t,x))\end{aligned}$$

- Using the notation $\widehat{f}(t) = \int_0^t f(s)ds$ for the primitive of f, we have

$$\begin{aligned}\gamma^\sigma(t,x) &= -\widehat{U_{xx}(t,x)x\kappa^*}(t,x) - U(t,x)\eta_t^\sigma \\ \gamma^\perp(t,x) &= \widehat{\nu}^{*,\perp}(t,U_x(t,x))\end{aligned}$$

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Dynamic Utilities with given optimal portfolio

Methodology : Let us start with a given optimal portfolio $X^*_.(x)$,

- In the classical utility optimization backward problem, He & Huang (1994) (in Markovian framework) try to characterize the terminal utility function, with a given optimal wealth X^* . Constraint on X^* also.
- Also interesting point of view of C.Rogers and co author.
- In the forward problem The problem is to diffuse the initial utility u using the information given by the path of X^*. Observe that :
  - The diffusion is not on u but on the derivative u_x
  - We have also to give the optimal state price density
- The only constraints are monotonicity of the both diffusion X^* and Y^* with respect to their initial condition or "equivalently" some Lispchitz condition on their coefficients

Desintegration results or Reverse Engineering

Main Result

- Assumption Assume the two equations admit monotonic solutions

$$
\begin{aligned}
dX^*_t &= X^*_t\left[r_t\,dt + \kappa^*_t(X^*_t).(dW_t + \eta^\sigma_t\,dt)\right] \\
dY^*_t &= -Y^*_t(x)\left[r_t\,dt + (-\nu^*_t(Y^*_t) + \eta^\sigma_t)dW_t\right]
\end{aligned}
$$

  with inverse processes $\mathscr{X}$ and $\mathscr{Y}$
- Assume that $u_{xx}(x)X^*_t(x)$ has a limit when x goes to infinity
- Construction Define the processes U and $\tilde{U}$ by

$$
U(t,x) = \int_0^x Y^*_t(u'(\mathcal{X}(t,z)))dz, \quad \tilde{U}(t,y) = \int_y^{+\infty} X^*_t(-\tilde{u}_y(\mathcal{Y}(t,z)))dz.
$$

- U is a $\mathscr{X}$-consistent stochastic utility satisfying the HJB type SPDE, and $\tilde{U}$ its $\mathscr{Y}$-consistent conjugate utility with optimal proc. X^* and Y^*.

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Linear optimal processes I

Assume that $X_t^*(x) = xX_t^*(1) = x\,X_t^*$ and $Y_t^*(y) = yX_t^*(1) = y\,Y_t^*$, and $Y_t^* X_t^*$ a true martingale.

- The optimal policies X_t^* and Y_t^* do not depend on x and y
- The inverse processes are $\mathcal{X}(t,z) = z/X_t^*$ and $\mathcal{Y}(t,z) = z/Y_t^*$
- For any consistent stochastic utility U, with initial utility u and linear optimal portfolios

$$U'_x(t, X_t^*(x)) = Y_t^*(u'_x(x)) \Longrightarrow U(t,x) = Y_t^* X_t^* u(x/X_t^*)$$

- There exists an equivalent martingale measure $\mathbb{Q}^*$, and a numeraire X_t^* such that in the new market $\widehat{U}(t,\widehat{x}) = Y_t^* X_t^* u(\widehat{x})$ is a $\mathbb{Q}^*$ dynamic consistent utility martingale.

Power Utilities

Power utility

- In particular if $u^{(\alpha)}(x) = \frac{x^{1-\alpha}}{1-\alpha}, \alpha \in]0,1]$, then $U^{(\alpha)}(t,x) = Z_t^{(\alpha)} u^{(\alpha)}(x)$ is a power utility with stochastic adjustment factor

$$Z_t^{(\alpha)} = Y_t^*(X_t^*)^\alpha$$

- There exists an optimal par (X_t^*, Y^*) with $(\alpha\kappa_t^* = -\eta_t^\sigma, \nu^* = 0)$ such that $Z_t^{(\alpha)}$ is decreasing.
- Markovian case : Markovian setting for the market with factors ξ_t, $Z_t^{(\alpha)} = \phi^{(\alpha)}(t, \xi_t)$

Backward calibration (He and Huang (1994) in the classical framework)

Find conditions to have $U^{(\alpha)}(T,x) = u^{(\alpha)}(x)$

- $Z_T^{(\alpha)} = 1 \Longrightarrow X_T^* = (Y_T^*)^{-1/\alpha}$
- The problem admits a solution only if $(Y_T^*)^{-1/\alpha}$ may be replicated by an

Utility ambiguity

Let us consider a agent with ambiguity on his utility ; he can made his choice in a family $\tilde{U}^\alpha$ of consistent stochastic utilities.

- α is a parameter with values in I, equipped with a priori probability measure $\mu(d\alpha)$
- the investor decides to allocate the initial wealth x into different initial wealths $x_\alpha(x)$ according to its anticipation, $x = \int_I x_\alpha(x)\mu(d\alpha)$
- he is looking for an optimal strategy as mixture of the individual optimal strategies

$$X_t^*(x) = \int_I X_t^{*,\alpha}(x_\alpha(x))\mu(d\alpha)$$

  monotone in x
- The main assumption is that the dual utility is a mixture of individual dual utilities

$$\tilde{U}^\mu(t,y) = \int_I \tilde{U}^\alpha(t,y)\mu(d\alpha)$$

Dual Mixture, and Sup-convolution

Dual Mixture

Let $\tilde{U}^\mu(t,y) = \int_I \tilde{U}^\alpha(t,y)\mu(d\alpha)$

- $\tilde{U}^\mu(t,y)$ is a convex consistent dual utilities with the same set of admissible state prices densities, if and only if there exists an admissible adjoint process $Y^*_t(y)$ optimal for all utilities $\tilde{U}^\alpha$
- Let $\tilde{u}^\mu(y) = \int_I \tilde{u}^\alpha(y)\mu(d\alpha)$ the dual initial utility. Then the optimal wealth of the primal problem is $X^*_t(x) = -\tilde{U}^\mu_y(t, Y^*_t(u^\mu_x(x))$. It is a mixture of optimal wealths

$$X^*_t(x) = \int_I X^{*,\alpha}_t(x_\alpha(x))\mu(d\alpha), \quad x_\alpha(x) = -\tilde{u}^\alpha_y(u_x(x))$$

- Sup-convolution result : Equilibrium, Pareto optimality,

Decreasing Utilities

- Applying this point of view to power decreasing utilities, we obtain one part of the beautiful result of Thaleia,and Rogers, that is the ”mixture” is still decreasing consistent utility,
- but not the fact there not other decreasing utility.
- Interpretation in terms of sup convolution may be interesting

Thank You for your attention !

Plan

Dynamic utilities from consumption and wealth I

- Extend the previous results to the case of dynamic utility functions to take into account that the preferences of the agent may changes with time.
- To get rid of the dependency on the maturity T.

- References : Musiela & Zariphopoulou, El Karoui & Mrad (dynamic utility functions from terminal wealth), Berrier & Rogers & Tehranchi (dynamic utility functions from consumption and terminal wealth).

Dynamic utility with consumption

Definition : Dynamic utility from consumption

- U^1 and U^2 two positive progressive utilities
- For all admissible wealth and consumption processes $(X^{x,c,\kappa}(.), c(.))$ satisfying $dX_t^{x,c,\kappa} = -c_t dt + dX_t^{x,\kappa}$ with $X_t^{x,c,\kappa} \geq 0$
  - $U^2(t, X_t^{x,c,\kappa}) + \int_0^t U^1(s, c_s)ds$ is a supermartingale.
  - There exists an optimal pair $(X^*(.), c^*(.))$ for which it is a martingale.

Dual structure A pair of conjugate dynamic utility functions

- For all state price density $Y^\nu(y)$, the following process is a submartingale : $\tilde{U}^2(t, Y_t^\nu(y)) + \int_0^t \tilde{U}^1(s, Y_s^\nu(y))ds.$
- Their exists an optimum ν^* with a martingale property
- $\tilde{U}^1(t, Y_t^{\nu^*}(y)) = c_t^*(y), \quad y = u_c^1(c_0^*)$

Plan

Motivations

Thanks to Isabelle Camilier, PhD

- Embedded long term interest rate risk in longevity-linked securities (maturity up to $30 - 50$ years.)
- Because of the lack of liquidity for long horizon, the standard financial point of view cannot be easily extended.
- Abundant literature on the economic aspects of long-term policy-making (Ekeland, Gollier, Weitzman...), often motivated by ecological issues (Hourcade & Lecocq)

The Ramsey Rule in Economics I

- Computation of a long term discount factor $R_0(T)$.
- A representative agent with :
  - u utility function
  - β pure time preference parameter
  - c aggregate consumption. Often a priori hypothesis are made on the form of the consumption function.
- Ramsey rule (link between consumption and discounting) :

$$R_0(T) = \beta - \frac{1}{T} \log \mathbb{E}\left[\frac{u'(c_T)}{u'(c_0)}\right].$$

(deduced from the maximization of the agent's intertemporal utility from consumption with infinite horizon)

The Ramsey Rule in Economics II

- Very popular particular case (Ramsey, 1928) :

$$R_0(T) = \beta + \gamma g,$$

  β pure time preference parameter, γ risk aversion, g growth rate.

- Example : Stern review on climate change (2006), with $\gamma = 1$, $g = 1.3\%$, $\beta = 0.1\% \to R_0(T) = 1.4\%$.
- Controversy between economists concerning parameters values.
  $R_0(T) = 1.4\%$: \$ 1 million in 100 years $\to$ \$ 250,000 today.
  $R_0(T) = 3.5\%$: \$ 1 million in 100 years $\to$ \$ 32,000 today.

Consumption Optimization and Ramsey rule

- Link between the state price density process and the marginal utility from consumption.

$$\frac{U^1_c(t, c^*_t)}{U^1_c(0, c^*_0)} = \frac{Y^*_t}{y} = \exp(-\int_0^t r_s ds) L^*_t(y)$$

where $L^*_t(y)$ is a change of proba measure with volatility $-\eta^\sigma_t + \nu^*_t(c^*_0)$

- Taking the expectation under the historical probability :

$$\mathbb{E}^{\mathbb{P}}\left[\frac{U^1_c(t,c^*_t)}{U^1_c(0,c^*_0)}\right] = \mathbb{E}^{\mathbb{P}}\left[\exp(-\int_0^t r_s ds) L^*_t(c^*_0)\right].$$

- In the classical Backward case, c^*_0 depends on the expectation of some function of the optimal path.
- In the forward case, c^*_0 does not depend on the future

Yields curve and Ramsey rule

- The yield curve The Ramsey rule is similar to price all zero-coupons (in incomplete market) using the Davis Rule that is under the optimal dual probability $L_t^*(y)$
- Acceptable for small trade
- for large trade use a second order correction term (indifference pricing ?)

Conclusion

- All Consistent dynamic utilities with a large degree of regularity continuous strictly may be generated as above.
- Valid for $(\mathcal{R}_t^\sigma(x),\ t \geq 0, x > 0)$ supposed only convex sets.
- Valid also for other classical optimization problem.
- Work in progress :
  - Progressive utility and consumption.
  - Model with jump.
  - Application to the state dependent utilities.

Ref Paper : El Karoui N. and M'rad M. : *An Exact Connection between two Solvable SDEs and a Non Linear Utility Stochastic PDEs* (2010-12) Arxiv.

Thank You for your attention !